

FIRST ORDER DEFORMATIONS OF PAIRS OF RATIONAL CURVES AND QUINTIC THREEFOLDS

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Abstract. This is one of a series of papers ([3], [4], [5]) that study the pairs of curves and hypersurfaces under the same motif: a fundamental property in the first order deformations of the pairs (the condition (1.2) below) has a deep effect on the structure of the normal sheaves of the curves in the hypersurfaces. Each paper solves an independent problem with the same condition (1.2), which addresses only one aspect of the normal sheaves. In this paper, we let X_0 be a specific type of smooth quintic threefolds (a generic quintic is this type) in projective space \mathbf{P}^4 over complex numbers, which is called a “totally non-degenerated” quintic threefold (see the definition (1.2)), and C_0 an irreducible rational curve on f_0 . We prove that if the first order deformation of the pair C_0, X_0 exists along each deformation of the hypersurface X_0 , i.e. the condition (1.2) below holds, then

$$H^1(N_{c_0/X_0}) = 0,$$

where $c_0 : \mathbf{P}^1 \rightarrow C_0$ is a normalization of C_0 .

1. Introduction to the deformation problem.

We consider the following specific condition on first order deformations of a pair. Let X_0 be a smooth quintic threefold in the projective space \mathbf{P}^4 of dimension 4 over the complex numbers. Let

$$c_0 : \mathbf{P}^1 \rightarrow C_0 \subset X_0$$

be the normalization map of an irreducible rational curve C_0 of degree d in X_0 . Let N_{c_0/X_0} be the normal bundle of the normalization which is uniquely determined by the exact sequence

$$0 \rightarrow T_{\mathbf{P}^1} \rightarrow c_0^*(T_{X_0}) \rightarrow N_{c_0/X_0} \rightarrow 0.$$

Also let N_{C_0/X_0} be the extension by zero to X_0 , of the normal sheaf $\mathcal{H}om(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_{C_0})$, where \mathcal{I} is the ideal sheaf of C_0 in X_0 .¹ Let

$$\mathbb{H}^1(T_{X_0} \rightarrow N_{C_0/X_0})$$

be the hypercohomology of the complex $T_{X_0} \rightarrow N_{C_0/X_0}$ that represents the tangent space to the deformations of the pair (C_0, X_0) , and $H^1(T_{X_0})$ be the space that represents the tangent space to the deformations of X_0 .

There is an exact sequence where the hypercohomology fits into,

$$(1.1) \quad \mathbb{H}^1(T_{X_0} \rightarrow N_{C_0/X_0}) \xrightarrow{\phi_1} H^1(T_{X_0}) \xrightarrow{\phi_2} H^1(N_{C_0/X_0})$$

¹In case C_0 is singular, the difference between the normal bundle N_{c_0/X_0} and the normal sheaf N_{C_0/X_0} does not affect this paper.

THEOREM 1.1. Assume X_0 is a “totally non-degenerated” quintic threefold, C_0 is an irreducible rational curve and c_0 is the normalization of C_0 . Then if

$$(1.2) \quad \phi_1 \text{ is surjective,}$$

$$(1.3) \quad H^1(N_{c_0/X_0}) = 0.$$

The totally non-degenerated quintic threefold $X_0 \subset \mathbf{P}^4$ is defined as follows. Throughout the paper, we always use the same letter $f_0 \in H^0(\mathcal{O}_{\mathbf{P}^4}(5))$ to denote its projectivization in

$$\mathbf{P}(H^0(\mathcal{O}_{\mathbf{P}^4}(5))),$$

and

$$X_0 = \text{div}(f_0).$$

Let \mathcal{Q} be the 5-dimensional subspace of $H^0(\mathcal{O}_{\mathbf{P}^4}(4))$, that consists of all the quartics in the form

$$\frac{\partial f_0}{\partial y},$$

where $y \in \mathbf{C}^5 = T_{\mathbf{C}^5}$ is a vector in \mathbf{C}^5 regarded as a tangent field. So \mathcal{Q} is the degree 4 subspace of the Jacobian ideal $J(f_0)$ of f_0 ,

$$\mathcal{Q} = (J(f_0))_4.$$

Because X_0 is smooth, \mathcal{Q} is an intrinsically defined, 5-d subspace in $H^0(\mathcal{O}_{\mathbf{P}^4}(4))$, where the number 5 comes from the dimension of the space $\{y\} = \mathbf{C}^5$.

Let $(u_0, u_1) \in (\mathbf{C}^5)^2$. We define a linear map $D(u_0, u_1, f_0)$:

$$(1.4) \quad \begin{array}{ccc} \mathcal{Q} & \xrightarrow{D(u_0, u_1, f_0)} & \mathbf{C}^5 \\ g & \rightarrow & (\frac{2^4}{0!}g(u_1), \dots, \frac{2^1}{3!}\frac{\partial^3 g(u_1)}{\partial^3 u_0}, \frac{2^0}{4!}\frac{\partial^4 g(u_1)}{\partial u_0^4}) \end{array}$$

DEFINITION 1.2. (**A totally non-degenerated quintic**).

Let $f_0 \in H^0(\mathcal{O}_{\mathbf{P}^4}(5))$ define a smooth quintic 3-fold X_0 . The quintic X_0 is called “totally non-degenerated”, if there is an $(u_0, u_1) \in (\mathbf{C}^5)^2$ such that $D(u_0, u_1, f_0)$ is non-degenerated.

PROPOSITION 1.3. The set of all totally non-degenerated quintics is a non empty, Zariski open set in the space of all smooth quintics in $H^0(\mathcal{O}_{\mathbf{P}^4}(5))$.²

Theorem 1.1 and proposition 1.3 solve the Clemens’ quintic conjecture.

² There is a question: are all smooth quintics totally non-degenerated?

1.1. A general approach-introduction to “blow-up” polynomials. The proof of proposition 1.3 is easy and self evident. We’ll leave it at the beginning of section 3. In the introduction we are only going to concentrate on the proof of theorem 1.1. The proof relies on the finding of an invariant, a 5×5 linear map $A(x, L, f_0)$ that hides rather deeply. Thus let’s slowly show the way into it. We begin the proof by considering the exact sequence

$$H^0(N_{c_0/\mathbf{P}^4}) \xrightarrow{\nu^s} H^0(c_0^*(N_{X_0/\mathbf{P}^4})) \simeq H^0(\mathcal{O}_{\mathbf{P}^1}(5d)) \rightarrow H^1(N_{c_0/X_0})$$

which is induced from the exact sequence of bundles,

$$(1.5) \quad 0 \rightarrow N_{c_0/X_0} \rightarrow N_{c_0/\mathbf{P}^4} \rightarrow c_0^*(N_{X_0/\mathbf{P}^4}) \rightarrow 0.$$

Then it is easy to see that $H^1(N_{c_0/X_0}) = 0$ is equivalent to the surjectivity of ν^s , i.e.

$$(1.6) \quad \text{Image}(\nu^s) = H^0(\mathcal{O}_{\mathbf{P}^1}(5d)).$$

DEFINITION 1.4. *We define*

$$B = \text{Image}(\nu^s).$$

Our proof of the surjectivity of ν^s , formula (1.6) is unconventional. We use only one element in B which is called a “blow-up” polynomial in the order $5d$. This is a turning point of the proof: switching the focus from the dimension of B to one specific element in B .

DEFINITION 1.5. *$\sigma \in B$ is called a blow-up polynomial in the order $5d - n$ at the set of zeros t_1, \dots, t_{5d-n} of σ if for any $s \in H^0(\mathcal{O}_{\mathbf{P}^1}(5d - n))$,*

$$sv \in B$$

where v is in $H^0(\mathcal{O}_{\mathbf{P}^1}(n))$ such that $\text{div}(v) = \text{div}(\sigma) - \sum_{i=1}^{5d-n} t_i$.

An immediate lemma is that

LEMMA 1.6. *If $\sigma \in B$ has distinct zeros and it is a blow-up polynomial in the order 1 at each zero of it, then σ is a blow-up polynomial in the order $5d$ at the set of all zeros of σ .*

Proof. If σ is a blow-up polynomial at a zero t_1 and at a zero t_2 ($t_1 \neq t_2$), by the linearity of B , it must be a blow-up polynomial in the order 2 at the set $\{t_1, t_2\}$. Then inductively, σ is a blow-up polynomial in the order $5d$ at the set of all zeros of σ . \square

The surjectivity of ν^s is equivalent to the existence of a “blow-up” polynomial in the order $5d$. By this lemma it is now equivalent to the existence of a “blow-up” polynomial in the order 1 at each zero of σ . However there are examples of C_0, X_0 where there are no “blow-up” polynomials in order $5d$. Thus formula (1.6) is not true

for all C_0, X_0 . This is where the condition (1.2) comes in. To do that, we consider another map ϕ^s :

$$(1.7) \quad \begin{array}{ccc} T_{f_0} \mathbf{P}^N & \xrightarrow{\phi^s} & H^0(\mathcal{O}_{\mathbf{P}^1}(5d)) \\ \alpha & \rightarrow & c_0^*(\frac{\partial F}{\partial \alpha}) \end{array}$$

where \mathbf{P}^N is the projectivization of $H^0(\mathcal{O}_{\mathbf{P}^4}(5))$ and F is a fixed universal quintic polynomial defining the universal quintic threefold,

$$\mathcal{X} = \{(x, f) \in \mathbf{P}^4 \times \mathbf{P}^N : f(x) = 0\}.$$

Next we use the key ingredient of the proof: the condition (1.2), i.e. $\phi_2 = 0$. If $\phi_2 = 0$ (or equivalently ϕ_1 is surjective), for any $\alpha \in T_{f_0} \mathbf{P}^N$, there is a $\langle \alpha \rangle \in H^0(c_0^*(T_{\mathbf{P}^4}))$ such that $(\alpha, \langle \alpha \rangle)$ is tangent to the universal quintic 3-fold \mathcal{X} in

$$\mathbf{P}^4 \times \mathbf{P}^N$$

i.e. $(\alpha, \langle \alpha \rangle)$ represents an element in hypercohomology

$$\mathbb{H}^1(T_{X_0} \rightarrow N_{C_0/X_0}).$$

$\langle \alpha \rangle$ is not unique, but we will fix a morphism

$$\alpha \rightarrow \langle \alpha \rangle$$

at one point $(c_0, f_0) \in M \times \mathbf{P}^N$. Because

$$c_0^*(\frac{\partial F}{\partial \alpha} + \frac{\partial f_0}{\partial \langle \alpha \rangle}) = 0,$$

$$image(\phi^s) \subset image(\nu^s).$$

There is another straightforward way to see this. We identify

$$(1.8) \quad T_{[f_0]} U_a = U_a,$$

where U_a is an affine open set of \mathbf{P}^N that contains $[f_0]$ as the origin. For any $f \in U_a$, \vec{f} denotes the vector corresponding to f in the isomorphism (1.8). Then the condition (1.2) simply says that

$$c_0^*(f)$$

lies in B because $c_0^*(f) = \phi^s(\vec{f})$.

Then this deformation condition (1.2) leads to our final theorem :

THEOREM 1.7. *Consider the quintic $x_0x_1x_2x_3x_4$ where*

$$x_i \in H^0(\mathcal{O}_{\mathbf{P}^4}(1)), i = 0, \dots, 4,$$

are generic. Then for the totally non-degenerated quintic X_0 , the condition (1.2) implies that $\phi^s(\overrightarrow{x_0x_1x_2x_3x_4})$ is a “blow-up” polynomial in the order $5d$ in B .

We found a specific “blow-up” polynomial in the order $5d$ in B . This completes the proof of theorem 1.1. Another expression for $\phi^s(\overrightarrow{x_0x_1x_2x_3x_4})$ is

$$\phi^s(\overrightarrow{x_0x_1x_2x_3x_4}) = c_0^*(x_0x_1x_2x_3x_4).$$

1.2. The proof of theorem 1.7—search for a “blow-up” polynomial in the order 5d. We still have not touched on the key invariant $A(x, L, f_0)$ mentioned before. We’ll introduce it in the proof theorem 1.7. The proof uses two invariants: one is a 5×5 linear map $A(x, L, f_0)$, the other is a subspace of the space of all quartics in \mathbf{P}^4 . For both invariants we assume $\phi_2 = 0$, i.e. ϕ_1 is surjective.

(1) The construction of the 5×5 linear map $A(x, L, f_0)$.

In the case $\phi_2 = 0$, we would like to construct a linear map $A(x, L, f_0)$ from $\mathcal{Q} \rightarrow \mathbf{C}^5$.

A special evaluation of quintics

Choose E to be a rank 2 subbundle of $c_0^*(T_{X_0})$ containing $c_0^*(T_{C_0})$. If N_{c_0/X_0} has a non-negative summand, E is reduced to this summand; if

$$N_{c_0/X_0} \simeq \mathcal{O}_{\mathbf{P}^1}(k_1) \oplus \mathcal{O}_{\mathbf{P}^1}(k_2)$$

has no non-negative summand, choose any E in which case E is not unique. Let $q \in \mathbf{P}^1$ be a point. Locally we can always have a decomposition

$$(1.9) \quad c_0^*(T_{X_0})|_{B_0} \simeq E|_{B_0} \oplus \mathcal{O}_{\mathbf{P}^1},$$

where B_0 is an open set of \mathbf{P}^1 containing the point q . For any $\beta \in c_0^*(T_{X_0})|_q$, $\beta|_q$ is the number in $\mathcal{O}_{\mathbf{P}^1}$ -summand in the decomposition (1.9). Assume ϕ_1 is surjective. Recall for any $\alpha \in T_{f_0}\mathbf{P}^N$, there is a $\alpha > \in H^0(c_0^*(T_{\mathbf{P}^4}))$ such that $(\alpha, \alpha >)$ is tangent to the universal quintic 3-fold \mathcal{X} in

$$\mathbf{P}^4 \times \mathbf{P}^N,$$

i.e. $(\alpha, \alpha >)$ represents an element in hypercohomology $\mathbb{H}^1(T_{X_0} \rightarrow N_{C_0/X_0})$. Then we have a special evaluation map

$$\begin{array}{ccc} T_{f_0}\mathbf{P}^N & \rightarrow & \mathbf{C} \\ \alpha & \rightarrow & \alpha > |_q. \end{array}$$

The evaluation $\alpha > |_q$ is not unique. But once f_0, q are fixed, we shall fix this evaluation.

Definition of the linear map $A(x, L, f_0)$.

Let $x = (x_0, \dots, x_4)$ be a coordinate’s system for \mathbf{C}^5 . Let W_1 be the line of $(\mathbf{C}^5)^*$ spanned by $(e_0)^* = (1, 0, 0, 0, 0)^*$, and W_4 spanned by

$$(e_1)^* = (0, 1, 0, \dots, 0)^*, \dots, (e_4)^* = (0, 0, \dots, 0, 1)^*,$$

where $(e_i)^*$ is the dual of e_i under the standard basis e_0, \dots, e_4 for \mathbf{C}^5 . Therefore

$$(\mathbf{C}^5)^* = W_1 \oplus W_4.$$

Then $\otimes_4(\mathbf{C}^5)^*$ has a direct-sum decomposition

$$(1.10) \quad \otimes_4(\mathbf{C}^5)^* = \left(\otimes_4 W_4 \right) \oplus \left(\otimes_3 W_4 \otimes W_1 \right) \oplus \dots \oplus \left(\otimes_4 W_1 \right).$$

Thus for any $g \in H^0(\mathcal{O}_{\mathbf{P}^4}(4))$, there is the decomposition in formula (1.10),

$$(1.11) \quad g = g_0 + g_1 \dots + g_4(x_0)^4,$$

where $g_i, i \neq 4$ are quartics, but g_4 is a complex number. Construct a linear map ψ_1 :

$$(1.12) \quad \begin{array}{ccc} H^0(\mathcal{O}_{\mathbf{P}^4}(4)) & \rightarrow & (\oplus_4 H^0(\mathcal{O}_{\mathbf{P}^4}(4)) \oplus \mathbf{C} \\ g & \rightarrow & (g_0, \dots, g_4). \end{array}$$

Let ψ_2 be the map

$$(1.13) \quad \begin{array}{ccc} (\oplus_4 H^0(\mathcal{O}_{\mathbf{P}^4}(4)) \oplus \mathbf{C} & \rightarrow & \mathbf{C}^5 \\ (g_0, \dots, g_4) & \rightarrow & (< Lg_0 >|_q, \dots, < Lg_3 >|_q, g_4) \end{array}$$

where $L \in H^0(\mathcal{O}_{\mathbf{P}^n}(1))$. Define

$$A(x, L, f_0) := \psi_2 \circ \psi_1|_{\mathcal{Q}}.$$

$A(x, L, f_0)$ has a 5×5 matrix representation if a basis for \mathcal{Q} is given. $A(x, L, f_0)$ is a linear map on the degree 4 subspace of the Jacobian ideal of f_0 . It depends on ϕ_1 and it isomorphically depends on the affine coordinates x .

(2) The construction of a subspace in $H^0(\mathcal{O}_{\mathbf{P}^4}(4))$.

For each fixed L, q with $L \in H^0(\mathcal{O}_{\mathbf{P}^n}(1))$, $L(c_0(q)) = 0$, let

$$V_{L,q} = \{Q \in H^0(\mathcal{O}_{\mathbf{P}^4}(4)) : < LQ >|_q \in E|_q\}.$$

Actually if $H^1(N_{C_0/X_0}) \neq 0$, $\mathcal{Q} \subset V_{L,q}$.

Now theorem 1.7 follows from the thread of the following propositions:

PROPOSITION 1.8. Assume $\phi_2 = 0$. If $V_{L,q} \neq H^0(\mathcal{O}_{\mathbf{P}^4}(4))$, and $X_0 = \text{div}(f_0)$ is totally non-degenerated, then for a fixed rank 2 bundle E above, a section L and $q \in \text{div}(L)$, there is a non-empty, open set of the space of all homogeneous coordinate's systems for \mathbf{P}^4 such that if x is in this open set, $A(x, L, f_0)$ is non-degenerated.

PROPOSITION 1.9. Assume $\phi_2 = 0$. If for a rank 2 bundle E above, a section L and $q \in \text{div}(L)$, there is a homogeneous coordinate's system x for \mathbf{P}^4 such that $A(x, L, f_0)$ is non-degenerated, then there are two cases: either

$$(a) \quad H^1(N_{c_0/X_0}) = 0$$

or

$$(b) \quad V_{L,q} = H^0(\mathcal{O}_{\mathbf{P}^4}(4)).$$

PROPOSITION 1.10. Assume $\phi_2 = 0$. If $V_{L,q} = H^0(\mathcal{O}_{\mathbf{P}^4}(4))$, then

$$\phi^s(\overrightarrow{x_1 x_2 \cdots x_5})$$

in theorem 1.7 is a "blow-up" polynomial in the order 5d.

Proposition 1.10 proves theorem 1.7. More precisely, if $\phi_2 = 0$, three propositions 1.8–1.10 together always lead to

$$H^1(N_{c_0/X_0}) = 0.$$

This proves theorem 1.1.

1.3. More explanation on the propositions 1.8–1.10 . The logic intertwined in the proposition 1.8–1.10 leads to a clear conclusion: if $\phi_2 = 0$, then

$$H^1(N_{c_0/X_0}) = 0.$$

Proposition 1.10 follows directly from the definitions. Proposition 1.9 is also straightforward which follows from the non-degeneracy of a Veronese variety. Even though they both have easy proofs, but these two propositions, especially the proposition 1.9 are crucial in understanding of the proof. They relate the vanishing of $H^1(N_{c_0/X_0})$ with the equality $V_{L,q} = H^0(\mathcal{O}_{\mathbf{P}^4}(4))$, and then with the degeneracy of $A(x, L, f_0)$. Proposition 1.8 is rather expected, but is the most difficult to prove. For instance, it is true for the generic point $([c_0], [f_0])$ on the Clemens-Katz component.³ Therefore proposition 1.8 reveals the deep truth about the non-degeneracy of $A(x, L, f_0)$, and propositions 1.9, 1.10 are relatively superficial assertions that reveal the contradiction of $H^1(N_{c_0/X_0}) \neq 0$ and the non-degeneracy of $A(x, L, f_0)$.

The proof of proposition 1.8 uses an important concept of “deformation to degeneracy”. This process includes two steps:

Step 1: Collapse the affine coordinates x ,

Step 2: Re-scale the map $A(x, L, f_0)$ so it becomes finite when the coordinates collapse.

But the objects as the x collapse are not unique. The objects in the limit of our degeneracy are collected in the following variety $\mathbf{P}(\mathcal{U})$: The subvariety $\mathbf{P}(\mathcal{U})$ in the product $\prod_2 \mathbf{P}^4 \times \prod_2 (\mathbf{P}^4)^*$ is defined by

$$\begin{aligned} \mathbf{P}(\mathcal{U}) = \{ & ([u_0], [u_1], [w_0], [w_1]) \in \prod_2 \mathbf{P}^4 \times \prod_2 (\mathbf{P}^4)^* : \\ & w_1(u_1) = w_1(u_0) = w_0(u_1) = 0, \} \end{aligned}$$

Let

$$\mathcal{U} \subset \prod_2 \mathbf{C}^5 \times \prod_2 (\mathbf{C}^5)^*$$

be the affine variety whose projectivization is $\mathbf{P}(\mathcal{U})$.

The linear maps behave in the following way during the collapsing. We let $D_{(w,L)}$ be the linear map from $\mathbf{C}^5 \rightarrow \mathbf{C}^5$ given by the diagonal matrix whose diagonal is (in the order)

$$\langle Lw_0^i w_1^{4-i} \rangle_{|q}, 1$$

where i is in order $i = 0, \dots, 3$ and $w \in \prod_2 (\mathbf{C}^5)^*$. Then we find another non-degenerated diagonal matrix D_x depending on coordinate's system x such that

$$D_x \circ A(x, L, f_0)$$

has a set of limits $D_{(w,L)} \circ D_{(u_0, u_1, f_0)}$ as the x coordinates collapse (depending on (w_0, w_1, u_0, u_1) in \mathcal{U}). Roughly speaking as the coordinate's vectors of x becomes

³ It was shown in [2] that by the combination of the arguments of Clemens and Katz, there is a component of the incidence scheme that dominates \mathbf{P}^N , where the rational curves in generic quintic threefolds are rigid.

linearly dependent in a certain way, all entries of $A(x, L, f_0)$ go to infinity and all entries of D_x go to zero, but their product goes to the finite linear map

$$D_{(w,L)} \circ D_{(u_0,u_1,f_0)}.$$

(re-scaling), where $(u_0, u_1, w) \in \mathcal{U}$. By the definition of totally non-degenerated quintic X_0 , for generic $(w, u) \in \mathcal{U}$,

$$D_{(w,L)} \circ D_{(u_0,u_1,f_0)}$$

is non-degenerated if $V_{L,q} \neq H^0(\mathcal{O}_{\mathbf{P}^4}(4))$. This non-degeneracy at the limit implies the non-degeneracy of $A(x, L, f_0)$ at generic x . This is the proof of proposition 1.8.

The logic through the proof of theorem 1.1 Once we get acquainted with the invariant $A(x, L, f_0)$, it is easy to see the logic in the proof. We always assume $\phi_2 = 0$ and X_0 is totally non-degenerated. Then

- (a) For each L , there is a generic x such that $A(x, L, f_0)$ is a non-degenerated 5×5 linear map (for example, on Clemens-Katz's component),
- (b) $H^1(N_{c_0/X_0}) \neq 0$ contradicts the non-degeneracy of $A(x, L, f_0)$ (for any coordinates x). Hence $H^1(N_{c_0/X_0}) = 0$.

The part (a) is proved by using the technique: “collapsing of affine coordinates”; The part (b) is proved by using the technique: “a blow-up polynomial”.

In the rest of paper, we are going to concentrate on proving theorem 1.1. In section 2, we give a description of the space B . The main point of this is to have more geometric criterion for “blow-up” polynomials. In section 3, we prove propositions 1.3, 1.9, 1.10. In section 4, we prove theorem 1.1

2. The image B and a description of “blow-up” polynomials .

In this section we describe B .

By the adjunction formula, we obtain the normal bundle

$$(2.1) \quad N_{c_0/X_0} \simeq \mathcal{O}_{\mathbf{P}^1}(k) \oplus \mathcal{O}_{\mathbf{P}^1}(-2-k),$$

where $k \geq -1$. This isomorphism is fixed throughout. Then it is clear that

$$H^1(N_{c_0/X_0}) = 0$$

if and only if $k = -1$. We always choose $E \subset c_0^*(T_{X_0})$ that is reduced to the summand $\mathcal{O}_{\mathbf{P}^1}(k)$.

PROPOSITION 2.1.

(1) *There are positive integers $d_1, d_2 \geq d$, $d_1 + d_2 = 5d - k - 2$ and*

$$\ell_1(t) \in H^0(\mathcal{O}_{\mathbf{P}^1}(5d - d_1)), \ell_2(t) \in H^0(\mathcal{O}_{\mathbf{P}^1}(5d - d_2))$$

such that

B

is the collection of polynomials

$$B = \{s_1\ell_1(t) + s_2\ell_2(t)\}$$

for all $s_i \in H^0(\mathcal{O}_{\mathbf{P}^1}(d_i))$.

(2) $(\nu^s)^{-1}(s_1\ell_1(t) + s_2\ell_2(t))$ lies in E at t_0 if and only if t_0 is a common zero of s_1, s_2 .

(3) A property of the “blow-up” polynomials:

Assume $H^1(N_{c_0/X_0}) \neq 0$ or equivalently $k \geq 0$. If $\omega(t) \in B$ is a blow-up polynomial at the zeros of $\omega_1(t)$ with $\omega_1 \in H^0(\mathcal{O}_{\mathbf{P}^1}(n))$, then any section in $(\nu^s)^{-1}(\omega)$ lies in E at the zeros of $\omega_1(t)$, and the vanishing order is the same as the multiplicity of $\omega_1(t)$. The converse is also true

Proof. : $\frac{c_0^*(T_{\mathbf{P}^4})}{E}$ is a rank 2-bundle over \mathbf{P}^1 and its degree is

$$c_1\left(c_0^*(T_{\mathbf{P}^4})\right) - c_1(E) = 5d - k - 2.$$

Thus it has decomposition

$$\frac{c_0^*(T_{\mathbf{P}^4})}{E} \simeq \mathcal{O}_{\mathbf{P}^1}(d_1) \oplus \mathcal{O}_{\mathbf{P}^1}(d_2),$$

with $d_1 + d_2 = 5d - k - 2$. Note there are obvious non-zero sections of $\frac{c_0^*(T_{\mathbf{P}^4})}{E}$ in the form

$$c_0^*(x_i \frac{\partial}{\partial x_j})$$

that vanish at d zeros of $c_0^*(x_i)$, where x_0, \dots, x_4 are homogeneous coordinates of \mathbf{P}^4 . And also because $\frac{c_0^*(T_{\mathbf{P}^4})}{E}$ is generated by global sections, d_i must be at least d . This shows the lower bound for d_i . On the other hand there is a vector bundle map $\bar{\nu}$

$$\begin{array}{ccc} \frac{c_0^*(T_{\mathbf{P}^4})}{E} & \rightarrow & c_0^*(N_{X_0/\mathbf{P}^4}) \\ \beta & \rightarrow & c_0^*(\frac{\partial f_0}{\partial \beta}). \end{array}$$

This map $\bar{\nu}$ must be in a form

$$(2.2) \quad \begin{array}{ccc} \mathcal{O}_{\mathbf{P}^1}(d_1) \oplus \mathcal{O}_{\mathbf{P}^1}(d_2) & \xrightarrow{\bar{\nu}} & \mathcal{O}_{\mathbf{P}^1}(5d) \\ (s_1, s_2)|_t & \rightarrow & s_1\ell_1(t) + s_2\ell_2(t), \end{array}$$

for some fixed $\ell_i(t) \in H^0(\mathcal{O}_{\mathbf{P}^1}(5d - d_i))$. Thus the map $\bar{\nu}^s$ on the sections

$$(2.3) \quad H^0\left(\frac{c_0^*(T_{\mathbf{P}^4})}{E}\right) \xrightarrow{\bar{\nu}^s} H^0(\mathcal{O}_{\mathbf{P}^1}(5d))$$

has an expression

$$\bar{\nu}^s(s_1, s_2) = \sum_{i=1}^2 s_i \ell_i$$

for $s_i \in H^0(\mathcal{O}_{\mathbf{P}^1}(d_i))$.

(2) It is clear (s_1, s_2) is zero at t_0 if and only if $s_i, i = 1, 2$ both vanish at t_0 . The proposition 2.1, part (2) is proved.

(3) This uses “base-point-free-pencil-trick”. In this part, $k \geq 0$. Assume ω has a factorization: $\omega = \omega_1 \omega_2$ where $\omega_1 \in H^0(\mathcal{O}_{\mathbf{P}^1}(n))$. Let

$$\eta_1 \in H^0(\mathcal{O}_{\mathbf{P}^1}(1))$$

be a linear factor of ω_1 , i.e. $\omega_1 = \omega'_1 \eta_1$. Choose any generic

$$\eta_2 \in H^0(\mathcal{O}_{\mathbf{P}^1}(1)).$$

Since $\eta_2 \omega'_1 \omega_2$ is also in B (because of the assumption), we may let

$$\eta_2 \omega'_1 \omega_2 = \lambda_1^2(t) \ell_1(t) + \lambda_2^2(t) \ell_2(t),$$

for some polynomials λ_i^2 . Also we have

$$\eta_1 \omega'_1 \omega_2 = \lambda_1^1(t) \ell_1(t) + \lambda_2^1(t) \ell_2(t),$$

for some other polynomials λ_i^1 . Hence

$$(\lambda_1^2(t) \eta_1 - \lambda_1^1(t) \eta_2) \ell_1(t) + (\lambda_2^2(t) \eta_1 - \lambda_2^1(t) \eta_2) \ell_2(t) = 0.$$

The degrees for the polynomials above are

$$(2.4) \quad \begin{aligned} \deg(\ell_1(t)) &= d_2 + k + 2, \deg(\ell_2(t)) = d_1 + k + 2 \\ \deg(\lambda_i^2) &= \deg(\lambda_i^1) = d_i, i = 1, 2. \end{aligned}$$

Then by the “base-point-free-pencil-trick” and the degrees for $k \geq 0$,

$$\lambda_1^2(t) \eta_1 - \lambda_1^1(t) \eta_2 = 0, \quad \lambda_2^2(t) \eta_1 - \lambda_2^1(t) \eta_2 = 0.$$

Since η_2 is generic, by the “base-point-free-pencil-trick” again,

$$\lambda_1^1(t), \lambda_2^1(t)$$

both vanish at the zero of η_1 . By the part (2),

$$(\nu^s)^{-1}(\omega)$$

lies in E at the zero of η_1 . Similarly,

$$(\nu^s)^{-1}(\omega)$$

lies in E at all zeros of ω_1 . This completes the proof. The converse follows from the part (2), then the part (1). \square

It is important to notice that this description of “blow-up” polynomials in part (3) is only valid in the case $H^1(N_{c_0/X_0}) \neq 0$. Otherwise any polynomial is a “blow-up” polynomial.

3. Application of blow-up polynomials .

In this section, we prove propositions 1.10, 1.9. The main purpose of these propositions are to convert original proof of $H^0(N_{c_0/X_0}) = 0$ to the proof of non-degeneracy of the linear map $A(x, L, f_0)$. The tool for the conversion is “blow-up polynomials”.

Before we do that let's prove proposition 1.3.

Proof. of proposition 1.3: Under the x coordinates of \mathbf{C}^5 , choose $u_0 = (1, 0, 0, \dots, 0)$, $u_1 = (0, 1, 0, \dots, 0)$ and a basis $\frac{\partial f_0}{\partial x_i}, i = 0, \dots, 4$ for \mathcal{Q} . Then $D(u_0, u_1, f_0)$ has a matrix expression

$$M(u_0, u_1) = \begin{pmatrix} \frac{16}{0!}a_{14000} & \frac{16}{0!}a_{05000} & \frac{16}{0!}a_{04100} & \frac{16}{0!}a_{04010} & \frac{16}{0!}a_{04001} \\ \frac{8}{1!}a_{23000} & \frac{8}{1!}a_{14000} & \frac{8}{1!}a_{13100} & \frac{8}{1!}a_{13010} & \frac{8}{1!}a_{13001} \\ \frac{4}{2!}a_{32000} & \frac{4}{2!}a_{23000} & \frac{4}{2!}a_{22100} & \frac{4}{2!}a_{22010} & \frac{4}{2!}a_{22001} \\ \frac{2}{3!}a_{41000} & \frac{2}{3!}a_{32000} & \frac{2}{3!}a_{31100} & \frac{2}{3!}a_{31010} & \frac{2}{3!}a_{31001} \\ \frac{1}{4!}a_{50000} & \frac{1}{4!}a_{41000} & \frac{1}{4!}a_{40100} & \frac{1}{4!}a_{40010} & \frac{1}{4!}a_{40001} \end{pmatrix}$$

where $a_{i_0 \dots i_4}$ are the coefficients of monomial $x_0^{i_0} \dots x_4^{i_4}$ in f_0 . The configuration of indexes $i_0 \dots i_4$ show the columns of the matrix are linearly independent for generic $a_{i_0 \dots i_4}$. Thus $\det(M(u_0, u_1))$ is a non-zero polynomial in $a_{i_0 \dots i_4}$. Then

$$\cup_{(u_0, u_1)} \left(H^0(\mathcal{O}_{\mathbf{P}^4}(5)) - \{ \det(M(u_0, u_1)) = 0 \} \right)$$

in the space of smooth quintics is the collection of all totally non-degenerated quintics. This completes the proof.

□

Proof. of proposition 1.10. If $H^1(N_{c_0/X_0}) = 0$, ν^s is surjective. Then all polynomials are blow-up polynomials in the order $5d$. So we assume $H^1(N_{c_0/X_0}) \neq 0$ or $k \geq 0$. Choose generic coordinates x_0, \dots, x_4 for \mathbf{C}^5 . Then $c_0^*(x_i), i = 0, \dots, 4$ would have $5d$ distinct zeros. Notice $\prod_i c_0^*(x_i)$ lies in B because it is just

$$\phi^s(\overrightarrow{x_0 \dots x_4})$$

where $\overrightarrow{x_0 \dots x_4}$ is the vector in $T_{f_0}\mathbf{P}^N$ that is the directional vector of the line through f_0 and $x_0 \dots x_4$. Applying the assumption

$$V_{L,q} = H^0(\mathcal{O}_{\mathbf{P}^4}(4)),$$

to

$$L = c_0^*(x_k), \quad \text{for each } k = 0, \dots, 4$$

and a zero q of $c_0^*(x_k)$, we obtain that

$$\overrightarrow{x_0 \dots x_4}$$

lies in E at each zero of $c_0^*(x_j), j = 0, \dots, 4$. Because $k \geq 0$, applying proposition 2.1, part (3), we obtain that the polynomial $\prod_i c_0^*(x_i)$ must a blow-up polynomial of degree 1 at each zero of

$$c_0^*(x_i), i = 0, \dots, 4.$$

By the lemma 1.6, $\prod_i c_0^*(x_i)$ must be a blow-up polynomial in order $5d$. We complete the proof. \square

PROPOSITION 3.1.

$V_{L,q}$ is a sub space in the linear space $H^0(\mathcal{O}_{\mathbf{P}^4}(4))$.

Proof. This directly follows from the definition of $V_{L,q}$. \square

Proof. of proposition 1.9. Choose any point $q \in \mathbf{P}^1$. Suppose

$$H^1(N_{c_0/X_0}) \neq 0.$$

Then by proposition 2.1, part 3,

$$\mathcal{Q} \subset V_{L,q} \subset H^0(\mathcal{O}_{\mathbf{P}^4}(4)).$$

Let $x = (x_0, \dots, x_4)$ be affine coordinates for \mathbf{C}^5 . Choose basis quartics $Q_j \in \mathcal{Q}, j = 0, \dots, 4$ for \mathcal{Q} , according to the decomposition (1.10), it can be expanded as

$$(3.1) \quad Q_j = (x_0)^4 \xi_4^j + (x_0)^3 \xi_3^j(\bar{x}) + (x_0)^2 \xi_2^j(\bar{x}) + x_0 \xi_1^j(\bar{x}) + \xi_0^j(\bar{x}),$$

where $\xi_k^j(\bar{x})$ is a homogeneous polynomial in $\bar{x} = (x_1, \dots, x_4)$ of degree $4 - k$. Since the linear map $A(x, L, f_0)$ is non-degenerated, we can choose five complex numbers $\epsilon_j, j = 0, \dots, 4$ such that

$$\sum_j \epsilon_j < L(x_0)^k \xi_k^j(\bar{x}) > |_q = 0$$

for each fixed $k = 3, 2, 1, 0$, but $\sum_j \epsilon_j \xi_4^j \neq 0$. Because

$$\sum_{j=0}^4 \epsilon_j Q_j \in V_{L,q},$$

x_0^4 must also be in $V_{L,q}$. Now consider the Veronese map v_4

$$(3.2) \quad \begin{array}{ccc} (\mathbf{P}^4)^* & \xrightarrow{v_4} & \mathbf{P}(H^0(\mathcal{O}_{\mathbf{P}^4}(4))) \\ x_0 & \rightarrow & x_0^4 \end{array}$$

whose image is the Veronese variety ([1], pg 25). Since Veronese variety is non-degenerated, $v_4((\mathbf{P}^4)^*)$ does not lie in any hyperplane. Thus

$$\text{span}(\{x_0^4\}_{(all \ x)}) = H^0(\mathcal{O}_{\mathbf{P}^4}(4)).$$

By proposition 3.1 (linearity of $V_{L,q}$) ,

$$\text{span}(\{x_0^4\}_{(all \ x)}) \subset V_{L,q}.$$

Therefore

$$V_{L,q} = H^0(\mathcal{O}_{\mathbf{P}^4}(4)).$$

We complete the proof. \square

4. Non-degeneracy of $A(x, L, f_0)$ –collapsing the affine coordinates. The central part of the proof is the construction of a deformation of affine coordinates for \mathbf{C}^5 to their degeneracy, i.e. collapsing of the affine coordinates. This is the idea of the proof: to show a limit of $A(x, L, f_0)$ after re-scaling, as x becomes linearly dependent, is non-degenerated.

Proof. of proposition 1.8. We assume all conditions for the proposition, in particular,

$$V_{L,q} \neq H^0(\mathcal{O}_{\mathbf{P}^4}(4)).$$

First let's express the linear map $A(x, L, f_0)$ as a 5×5 matrix. Let

$$y = [y_0, \dots, y_4]$$

be homogeneous coordinates for \mathbf{P}^4 . Define the five quartics

$$Q_j = \frac{\partial f_0}{\partial y_j}, j = 0, \dots, 4,$$

that form a basis of \mathcal{Q} .

We'll fixed y coordinate's system. So five quartics $Q_j, j = 0, \dots, 4$ are also fixed. Under the coordinates x and y for \mathbf{C}^5 , the linear map $A(x, L, f_0)$ has a matrix representation

$$\mathcal{A}_{(L,x,y)} = \begin{pmatrix} \xi_4^0 & \xi_4^1 & \dots & \dots & \xi_4^4 \\ \langle LQ_0(\bar{x}) \rangle|_q & \langle LQ_1(\bar{x}) \rangle|_q & \dots & \dots & \langle LQ_4(\bar{x}) \rangle|_q \\ \langle L\frac{\partial Q_0(\bar{x})}{\partial x_0}x_0 \rangle|_q & \langle L\frac{\partial Q_1(\bar{x})}{\partial x_0}x_0 \rangle|_q & \dots & \dots & \langle L\frac{\partial Q_4(\bar{x})}{\partial x_0}x_0 \rangle|_q \\ \langle L\frac{\partial^2 Q_0(\bar{x})}{2\partial x_0^2}x_0^2 \rangle|_q & \langle L\frac{\partial^2 Q_1(\bar{x})}{2\partial x_0^2}x_0^2 \rangle|_q & \dots & \dots & \langle L\frac{\partial^2 Q_4(\bar{x})}{2\partial x_0^2}x_0^2 \rangle|_q \\ \langle L\frac{\partial^3 Q_0(\bar{x})}{6\partial x_0^3}x_0^3 \rangle|_q & \langle L\frac{\partial^3 Q_1(\bar{x})}{6\partial x_0^3}x_0^3 \rangle|_q & \dots & \dots & \langle L\frac{\partial^3 Q_4(\bar{x})}{6\partial x_0^3}x_0^3 \rangle|_q \end{pmatrix}$$

Thus it suffices to prove that the 5×5 matrix $\mathcal{A}_{(L,x,y)}$ is non-degenerated.

The first row in \mathcal{A} is just

$$\xi_4^j = \frac{\partial^5 f_0}{4! \partial y_j \partial x_0^4}, j = 0, \dots, 4,$$

that only depends on the $\frac{\partial}{\partial x_0}, \frac{\partial}{\partial y_j}$ and the quintic X_0 . The determinant of the matrix $\mathcal{A}_{(L,x,y)}$ continuously depends on the tangent vectors

$$\overrightarrow{L\frac{\partial^i Q_j(\bar{x})}{i! \partial x_0^i}x_0^i} \in T_{f_0} \mathbf{P}^N,$$

where $L\frac{\partial^i Q_j(\bar{x})}{i! \partial x_0^i}x_0^i \in H^0(\mathcal{O}_{\mathbf{P}^4}(5))$ are quintics, and \overrightarrow{f} is a directional vector of the line through the quintic f and f_0 . Thus it suffices to prove the non-degeneracy of $\mathcal{A}_{(L,x,y)}$ under any one of the pairs of coordinates x, y . In order to do that, we collapse the x coordinates. So we would like to know a projective limit of $\mathcal{A}_{(L,x,y)}$ as x coordinates collapse or degenerate. The following is how we proceed it. We construct a continuous family of the matrices $\mathcal{B}_{(L,\theta,y)}$ where θ is in the irreducible variety \mathcal{V}_1 , such that the generic member $\mathcal{B}_{(L,\theta,y)}$ in this family is a product of a non-degenerated diagonal matrix and the matrix $\mathcal{A}_{(L,x,y)}$, where the x, y coordinates are generic. The advantage

of this family $\mathcal{B}_{(L,\theta,y)}$ is that it has a natural compactification and the matrices $\mathcal{B}_{(L,\theta,y)}$ over some boundary points (L, θ_0, y) are finite and computable. This is not the case for $\mathcal{A}_{(L,x,y)}$, which has no obvious (natural) compactification and is not computable. (the quintic X_0 is not moving in both families, but the tangent vectors of X_0 are deforming to “degenerated” positions as x coordinates collapse).

Let’s describe the family $\mathcal{B}_{(L,\theta,y)}$ in an intrinsic manner to see the collapsing of x - coordinates. Let $u_0 \in \mathbf{C}^5$. Then $u_0 \in T_{any\ point}(\mathbf{C}^5)$ and Q_j is a function on \mathbf{C}^5 . Thus the i th derivative of Q_j

$$\frac{\partial^i Q_j}{\partial u_0^i} \quad (u_0 \text{ is a vector field}).$$

lies in

$$H^0(\mathcal{O}_{\mathbf{P}^4}(4-i)) = Sym^{4-i}((\mathbf{C}^5)^*),$$

where Sym^* is the symmetric algebra of a vector space. Let $s \in \mathbf{C}^5 \otimes (\mathbf{C}^5)^*$. Let η_1^s be the linear map

$$(4.1) \quad \begin{array}{ccc} (\mathbf{C}^5)^* & \xrightarrow{\eta_1^s} & (\mathbf{C}^5)^* \\ \alpha & \rightarrow & \alpha(s) \end{array}$$

where $\alpha(u \otimes w) = \alpha(u)w$ for $s = u \otimes w$. By the universal bilinear property of the tensor product, η_1^s is well-defined. Then we obtain another map η_2^s :

$$(4.2) \quad \begin{array}{ccc} Sym^{4-i}((\mathbf{C}^5)^*) & \xrightarrow{\eta_2^s} & Sym^{4-i}((\mathbf{C}^5)^*) \\ h(\sigma_1, \dots, \sigma_{4-i}) & \rightarrow & h(\eta_1^s(\sigma_1), \dots, \eta_1^s(\sigma_{4-i})) \end{array},$$

where $\sigma_j \in (\mathbf{C}^5)^*$. Let $L \in H^0(\mathcal{O}_{\mathbf{P}^4}(1))$, $w_0 \in H^0(\mathcal{O}_{\mathbf{P}^4}(1))$. Using formula (4.2), we obtain a family of quintics $g_{(L,s,y,w_0)}$,

$$(4.3) \quad g_{(L,s,y,w_0)} = L \eta_2^s \left(\frac{\partial^i Q_j}{i! \partial u_0^i} \right) w_0^i$$

that is parametrized by L, s, y, w_0 . But the matrix constructed from the evaluations of these quintics is not always the desired matrix $\mathcal{A}_{(L,x,y)}$ because $s \in \mathbf{C}^5 \otimes (\mathbf{C}^5)^*$ is too arbitrary. We need to limit s to a smaller set. Let

$$(u_0, \dots, u_4) \in \prod_5 \mathbf{C}^5, (w_0, \dots, w_4) \in \prod_5 (\mathbf{C}^5)^*$$

be the coordinates. Later in the proof we’ll use $u_i, w_i, i = 0, \dots, 4$ to indicate which copy of \mathbf{C}^5 or $(\mathbf{C}^5)^*$ we are referring to. Let

$$\mathcal{V}_1 \subset \prod_5 \mathbf{C}^5 \times \prod_5 (\mathbf{C}^5)^*$$

be the sub-scheme defined by

$$(4.4) \quad \begin{array}{l} w_i(u_j) = 0, \text{ for all } i \neq j \\ w_i(u_i) - w_j(u_j) = 0, \text{ for } i \neq 0, j \neq 0. \end{array}$$

LEMMA 4.1. \mathcal{V}_1 is irreducible.

The proof of which will be deferred to the end of this section.

Let

$$(4.5) \quad \mathcal{V}_2 = \left\{ \sum_{i=1}^4 u_i \otimes w_i : \theta = ((u_0, \dots, u_4), (w_0, \dots, w_4)) \in \mathcal{V}_1 \right\}$$

contained in $\mathbf{C}^5 \otimes (\mathbf{C}^5)^*$. Denote the elements in \mathcal{V}_2 by $p(\theta), \theta \in \mathcal{V}_1$ (\mathcal{V}_2 is the collection of the desired s). Now finally we obtain that the family (for each fixed i, j),

$$(4.6) \quad q_{(L, \theta, y, i, j)} = L \eta_2^{p(\theta)} \left(\frac{\partial^i Q_j}{i! \partial u_0^i} \right) (w_0)^i, i = 0, \dots, 3, j = 0, \dots, 4.$$

is a flat family of reducible quintics⁴ over

$$(4.7) \quad \Theta = H^0(\mathcal{O}_{\mathbf{P}^4}(1)) \times \mathcal{V}_1 \times \prod_5 \mathbf{C}^5,$$

where

$$L \in H^0(\mathcal{O}_{\mathbf{P}^4}(1)), \theta \in \mathcal{V}_1, y \in \prod_5 \mathbf{C}^5.$$

It is important to notice that the parameter space Θ is irreducible.

Now going back to the original extrinsic matrix $\mathcal{A}_{(L, x, y)}$, we can construct x coordinates to recover $\mathcal{A}_{(L, x, y)}$ (depending on the coordinates) in the following way: Given a GENERIC $\theta \in \mathcal{V}_1$, $\theta = ((u_0, \dots, u_4), (w_0, \dots, w_4))$, we have a basis $\{u_0, \dots, u_4\}$ for \mathbf{C}^5 and a basis $\{w_0, \dots, w_4\}$ for $(\mathbf{C}^5)^*$ satisfying

$$(4.8) \quad \begin{aligned} w_i(u_j) &= 0, i \neq j \\ w_i(u_i) &= l \neq 0, i \neq 0 \\ w_0(u_0) &\neq 0. \end{aligned}$$

With the basis u_0, \dots, u_4 , we construct the matrix $\mathcal{A}_{L, x_u, y}$, where x_u are the coordinates under the basis u . The entries of it, by the definition, are

$$(4.9) \quad < L \frac{\partial^i Q_j(0, x_1, \dots, x_4)}{i! \partial u_0^i} x_0^i > |_q$$

where $x_u = (x_0, x_1, \dots, x_4)$ are the coordinates for \mathbf{C}^5 under the basis u . If we regard the coordinates $x_i \in (\mathbf{C}^5)^*$, then by the definition of \mathcal{V}_1 ,

$$(4.10) \quad x_i = \frac{w_i}{w_i(u_i)}, i = 0, \dots, 4.$$

⁴ We only need the map

$$\mathcal{V}_1 \rightarrow H^0(\mathcal{O}_{\mathbf{P}^4}(5))$$

to be continuous. But the family as a family of hypersurfaces is automatically flat.

Plugging formula (4.10) into formula (4.9), we obtain

$$(4.11) \quad L \frac{\partial^i Q_j((0, x_1, \dots, x_4))}{i! \partial u_0^i} x_0^i = \frac{1}{l^{4-i}(w_0(u_0))^i} q_{(L, \theta, y, i, j)}.$$

(see the formula (4.8) for l). Thus the entries of $\mathcal{A}_{L, x_u, y}$ are

$$(4.12) \quad < \frac{1}{l^{4-i}(w_0(u_0))^i} q_{(L, \theta, y, i, j)} > |_q.$$

Let

$$(4.13) \quad b_{ij} = < q_{(L, \theta, y, i, j)} > |_q.$$

and

$$(4.14) \quad \mathcal{B}_{(L, \theta, y)} = \begin{pmatrix} \xi_4^0 & \xi_4^1 & \dots & \dots & \xi_4^4 \\ b_{00} & b_{01} & \dots & \dots & b_{04} \\ b_{10} & b_{11} & \dots & \dots & b_{14} \\ b_{20} & b_{21} & \dots & \dots & b_{24} \\ b_{30} & b_{31} & \dots & \dots & b_{34} \end{pmatrix}$$

We obtain

$$(4.15) \quad \det(\mathcal{B}_{(L, \theta, y)}) = l^{10} (w_0(u_0))^6 \det(\mathcal{A}_{L, x_u, y}).$$

(The $\det(\mathcal{A}_{L, x_u, y})$ is infinite if one of $w_i(u_i)$ is zero). It suffices to show $\det(\mathcal{B}_{(L, \theta, y)}) \neq 0$ for generic $\theta \in \mathcal{V}_1$. To show this, we consider a non-generic member in this family $\mathcal{B}_{(L, \theta, y)}$ when the u -coordinates collapse (some $w_i(u_i) = 0$). We define the subvariety $\mathbf{P}(\mathcal{U})$ in the product $\prod_2 \mathbf{P}^4 \times \prod_2 (\mathbf{P}^4)^*$ to be

$$\mathbf{P}(\mathcal{U}) = \{([u_0], [u_1], [w_0], [w_1]) \in \prod_2 \mathbf{P}^4 \times \prod_2 (\mathbf{P}^4)^* : \\ w_1(u_1) = w_1(u_0) = w_0(u_1) = 0, \}$$

Note $\mathbf{P}(\mathcal{U})$ is irreducible. Let

$$\mathcal{U} \subset \prod_2 \mathbf{C}^5 \times \prod_2 (\mathbf{C}^5)^*$$

be the affine variety whose projectivization is $\mathbf{P}(\mathcal{U})$. Now $\mathbf{P}(\mathcal{U})$ collects non-generic members of \mathcal{V}_2 in the following way: Choose a generic

$$(4.16) \quad ((u_0, u_1), (w_0, w_1)) \in \mathcal{U}.$$

Let $\theta_0 = ((u_0, u_1, u_1, 0, 0), (w_0, w_1, w_1, w_3, w_4)) \in \mathcal{V}_1$. Then

$$p(\theta_0) = 2u_1 \otimes w_1 \in \mathcal{V}_2.$$

Then plugging $p(\theta_0) = 2u_1 \otimes w_1$ to formula (4.6), we obtain that

$$(4.17) \quad q_{(\theta_0, L, y, i, j)} = L \frac{2^{4-i} \partial^i Q_j(u_1)}{i! \partial u_0^i} (w_1)^{4-i} (w_0)^i$$

is a section of

$$H^0(\mathcal{O}_{\mathbf{P}^4}(5)).$$

(notice the quintics $q_{(\theta_0, L, y, i, j)}$ are monomials now). As the generic θ goes to the degenerated

$$\theta_0$$

the quintic $q_{(L, \theta, y, i, j)}$ continuously (as points) goes to the monomial

$$L \frac{2^{4-i} \partial^i Q_j(u_1)}{i! \partial u_0^i} (w_1)^{4-i} (w_0)^i.$$

Thus the continuous limit of $\mathcal{B}_{L, \theta, y}$ is

$$\mathcal{B}(L, \theta_0, y) = D(w, L) \bar{D}(u_0, u_1, f_0),$$

where $\bar{D}(u_0, u_1, f_0)$ is a 5×5 matrix whose first row is

$$(4.18) \quad \frac{\partial^5 f_0}{4! \partial y_j \partial u_0^4}, j = 0, 1, 2, 3, 4,$$

and k -th row is

$$(4.19) \quad 2^{6-k} \frac{\partial^5 f_0(u_1)}{(k-2)! \partial y_j \partial u_0^{k-2}}, j = 0, 1, 2, 3, 4.$$

for $k = 2, 3, 4, 5$, and $D(w, L)$ is the diagonal matrix defined in section 1.3. If y is a coordinates' system for \mathbf{C}^5 , $\bar{D}(u_0, u_1, f_0)$ is a matrix representation of $D(u_0, u_1, f_0)$. Next notice X_0 is totally non-degenerated. There are y coordinates such that $\bar{D}(u_0, u_1, f_0)$ is non-degenerated. Next we show $D(w, L)$ is non-degenerated. By its definition, it suffices to show

$$< L(w_0)^i (w_1)^{4-i} > |_q \neq 0$$

for each $i = 3, 2, 1, 0$. Notice the projection from \mathcal{U} to $\prod_2(\mathbf{C}^5)^*$ is surjective. Then the genericity of $(u, w) \in \mathcal{U}$ implies the genericity of $w \in \prod_2(\mathbf{C}^5)^*$. Using our assumption

$$V_{L, q} \neq H^0(\mathcal{O}_{\mathbf{P}^4}(4))$$

it suffices to prove

$$(4.20) \quad \text{span}(\{(w_0)^i (w_1)^{4-i}\}_{w \in \prod_2(\mathbf{C}^5)^*}) = H^0(\mathcal{O}_{\mathbf{P}^4}(4))$$

for each i .

Consider $(w_0)^2 (w_1)^2$. Notice the map

$$(4.21) \quad \begin{array}{ccc} v_{2,2} : \prod_2(\mathbf{C}^5)^* & \xrightarrow{v_{2,2}} & \mathbf{P}(H^0(\mathcal{O}_{\mathbf{P}^4}(4))) \\ (w_0, w_1) & \rightarrow & (w_0)^2 (w_1)^2 \end{array}$$

is the composition of the Veronese map and the Segre map (see [1]). By the definitions of these maps, its composition yields the non-degenerated image in

$$H^0(\mathcal{O}_{\mathbf{P}^4}(4)).$$

This proves formula (4.20).

Similarly, using exactly the same argument with the Veronese maps and the Segre maps as above, we prove that

$$\text{span}(\{(w_0)^i(w_1)^{4-i}\}_{w \in \Pi_2(\mathbf{C}^5)^*}) = H^0(\mathcal{O}_{\mathbf{P}^4}(4))$$

for all the other $i = 0, 1, 3$.

This proves that $D(w, L)$ is non-degenerated. Thus

$$\mathcal{B}_{(L, \theta_0, y)}$$

is non-degenerated.

By deforming (L, θ_0, y) to a generic (L, θ, y) in the irreducible

$$(4.22) \quad \begin{aligned} & \{L\} \times \mathcal{V}_1 \times \prod_5 \mathbf{C}^5 \subset \Theta \\ & (L \text{ is fixed}), \end{aligned}$$

we obtain that

$$\mathcal{B}_{(L, \theta, y)},$$

is non-degenerated. (Because \mathcal{V}_1 is irreducible). Then by the formula (4.15),

$$\det(\mathcal{A}_{(L, x_u, y)}) \neq 0.$$

Since u is a generic basis for \mathbf{C}^5 , we proved the proposition. \square

We conclude the paper with an easy proof of lemma 4.1.

Proof. of lemma 4.1. Fix bases $\{\alpha_i\}$ for \mathbf{C}^5 and $\{\beta_i\}$ for $(\mathbf{C}^5)^*$ such that $\beta_i(\alpha_j) = 0$ for $i \neq j$ and $\beta_i(\alpha_i) = 1$. Then $\alpha_i \rightarrow \beta_i$ gives an isomorphism

$$\mathbf{C}^5 \simeq (\mathbf{C}^5)^*.$$

Let $\mathcal{V}_0 \subset \prod_5(\mathbf{C}^5) \times \prod_5(\mathbf{C}^5)^*$ be the scheme defined by

$$(4.23) \quad \begin{aligned} & w_i(u_j) = 0, \text{ for all } i \neq j \\ & w_i(u_i) - w_j(u_j) = 0, \text{ for all } i, j. \end{aligned}$$

where

$$((u_0, \dots, u_4), (w_0, \dots, w_4)) \in \prod_5(\mathbf{C}^5) \times \prod_5(\mathbf{C}^5)^*.$$

Then

$$\mathcal{V}_1 \cap \{w_0(u_0) = w_1(u_1)\} = \mathcal{V}_0.$$

Thus it suffices to show \mathcal{V}_0 is irreducible. Let

$$U = \{((u_0, \dots, u_4), z(u_0, \dots, u_4)) : u_i \text{ are linearly independent}\}$$

be the subset of $\prod_5(\mathbf{C}^5) \times \prod_5(\mathbf{C}^5)^*$, where z is a non-zero constant and $z(u_0, \dots, u_4)$ is an element in $\prod_5(\mathbf{C}^5)^*$ via the isomorphism above. Then the closure \bar{U} is an irreducible variety. It is obvious that

$$\bar{U} \subset \mathcal{V}_0.$$

Let g be any polynomial not in the ideal $I_{\mathcal{V}_0}$. It is also obvious that

$$\{g = 0\} \cap \mathcal{V}_0 \not\supset U.$$

Thus $\bar{U} = \mathcal{V}_0$ is irreducible.

□

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